Analysis of Two-Dimensional Steady-State Heat Transfer in a Rectangular Region

C. L. Ko*
Oakland University, Rochester, Michigan
and
C. W. Bert†
University of Oklahoma, Norman, Oklahoma

The objective of this investigation is to apply a new version of the Rayleigh-Ritz method, called the Rayleigh-Schmidt method (RSM), to a two-dimensional heat conduction problem involving a rectangular region with convective boundary conditions on three sides and with or without internal heat generation. Numerical results obtained by using this new technique are compared with those obtained by the conventional Rayleigh-Ritz method (RRM), the finite-difference technique (FDM), the method of Fourier transform (MFT), and the heat balance integral method (HBIM). For the case of a rectangular region in the absence of internal heat generation, the results are also compared with those obtained from an exact solution presented for the first time in this paper. These comparisons indicate that the RSM gives reasonably accurate results in all cases, but that the HBIM is not very accurate when there is internal heat generation.

Nomenclature

a_x, a_y = $k_x T_f/L^2$, $k_y T_f/b^2$
B_x, B_y = $h_x L/k_x$, $h_y L/k_y$
b = half width of region; see Fig. 1
C_x, C_y = $a_x B_x$, $a_y B_y$
F = function defined in Eq. (D7)
g = internal heat generation rate per unit volume
h_x, h_y = convective heat transfer coefficients on surfaces $X = L$ and $Y = \pm b$
I = functional defined in Eq. (23)
k_x, k_y = thermal conductivities in the X and Y directions
L = length of region; see Fig. 1
$\dot{Q}$ = heat transfer rate
T, T_b = temperature in general and at the base
T_f = bulk temperature of the fluid surrounding the region
T_s = temperature at the corner of base ($x = 0$, $y = 1$)
X, Y = rectangular coordinates; see Fig. 1
x, y = X/L, Y/b
α, β = parameters defined in Eqs. (C4)
γ = $B_y/2$
θ = $(T - T_f)/T_f$
θ_0, θ_s = θ values along edge ($y = 1$) and at corner of base ($x = 0$, $y = 1$)
θ_t = θ value at tip ($x = 1$, $y = 1$)
μ_m, ν_n = mth and nth roots of Eqs. (D8) and (D9)
ϕ = $\theta - \theta_b$

Introduction

THE exact solution of the two-dimensional steady-state conduction problem of a rectangular region with convective boundary conditions in both directions has never been reported in the literature. Sfeir[1] applied the heat balance integral method (HBIM) to this problem and obtained satisfactory agreement with a finite-difference solution. The HBIM was first introduced by Goodman,[2] and the concept was taken from the technique of solving the integral boundary-layer equations given by Pohlhausen and von Karman.[3] To apply this technqiue, one is first required to assume a trial function of one position variable. Then, one must solve the relations of variables at boundaries that are derived by integrating the governing equation with respect to the same position variable. The trial function is normally assumed to be polynomial, which does not necessarily satisfy the governing equation. Therefore, the solution obtained may be far from exact. Hence, one of the objectives of this investigation is to develop an exact solution for the governing equation of two-dimensional heat conduction with convective boundary conditions. Comparing its results with approximate solutions obtained by various methods for the same subject problem, one can then attain reasonable feeling regarding the accuracy of different approximate techniques.

For the case of a two-dimensional rectangular region with internal heat generation, the governing equation becomes nonhomogeneous and to obtain its exact solution becomes practically infeasible. Thus, approximate techniques become the only means of analysis. One of the well-known approximate techniques is the Rayleigh-Ritz method.[4] To apply this technique, the trial function is generally assumed to be a polynomial with integer power. However, it is not necessary to use integer powers, as was shown by Rayleigh[5] and Temple and Bickley.[6] The trial function can contain a power-law term with an undetermined exponent. Schmidt[7] and, more recently, many others applied this version of the Rayleigh-Ritz method to various structures problems. See Ref. 8 for an extensive survey. Recently, Bert[9] also applied this technique, called the Rayleigh-Schmidt method (RSM), to a one-dimensional and a simple two-dimensional problem of steady heat conduction. Laura et al.[10] also used the RSM to calculate the temperature distributions of a two-dimensional region with internal heat generation and homogeneous Dirichlet boundary conditions. To this date, it has not been applied successfully to a more general two-dimensional case with both internal heat generation and convective boundary conditions. Therefore, an objective of this investigation is to evaluate the applicability of the RSM to this problem by comparing its results with various existing approximate methods.

Received Dec. 18, 1986; revision received June 30, 1987.

*Assistant Professor, Department of Mechanical Engineering. Member AIAA.

†Parkinson Chair Professor of Engineering, School of Aerospace, Mechanical, and Nuclear Engineering. Associate Fellow AIAA.

Formulation and Exact Solution for a Rectangular Region

The governing equation of the steady-state temperature distribution in a two-dimensional rectangular region shown in Fig. 1 can be normalized and expressed as follows:

$$\frac{\partial}{\partial x}\left(a_x \frac{\partial \theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_y \frac{\partial \theta}{\partial y}\right)=0 \tag{1}$$

Since the temperature distribution is symmetric about the x axis, one can determine the boundary conditions as

$$\theta|_{x=0}=\theta_b(y) \tag{2}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=1}=-B_x\theta|_{x=1} \tag{3}$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=0}=0 \tag{4}$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=1}=-B_y\theta|_{y=1} \tag{5}$$

The base temperature T_b is a function of y in general.

Using the method of separation of variables and taking a_x and a_y as constants, one can obtain the solution of Eq. (1) that satisfies conditions (3), (4), and (5) for the case of constant a_x and a_y as

$$\theta=\sum_{m=1}^{\infty} A_m \cos\alpha_m y\,(e^{p\alpha_m x}+\beta_m e^{-p\alpha_m x}) \tag{6}$$

where

$$p\equiv(a_y/a_x)^{1/2}, \qquad \beta_m\equiv\frac{(p\alpha_m+B_x)}{(p\alpha_m-B_x)}\,e^{2p\alpha_m}$$

and α_m is the mth zero of the equation

$$\alpha \tan\alpha=B_y \tag{7}$$

It is obvious that the negated value of a root of Eq. (7) is itself also a root of the same equation. Therefore, by restricting values of α_m to be positive, one can rewrite Eq. (6) as

$$\theta=\sum_{m=1}^{\infty}\left[A_m(e^{p\alpha_m x}+\beta_m e^{-p\alpha_m x}) + A_m^*\left(e^{-p\alpha_m x}+\frac{1}{\beta_m}e^{p\alpha_m x}\right)\right]\cos\alpha_m y \tag{8}$$

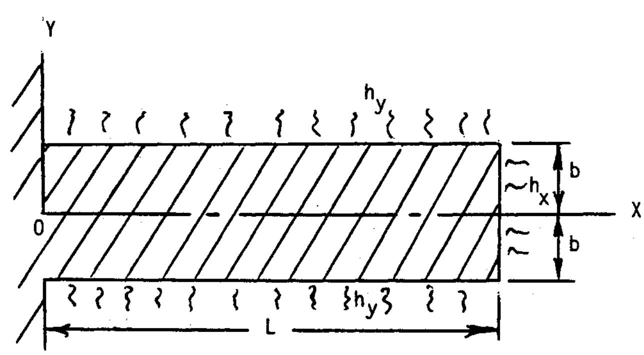

Fig. 1 A two-dimensional region with convective heat transfer at three edges.

where A_m^* is the coefficient corresponding to the negative root $-\alpha_m$. The coefficient A_m^* should not be discarded because the corresponding eigenfunction of x is different from those associated with A_m. Since the eigenvalues α_m and $-\alpha_m$ yield the same eigenfunction of y, $\cos\alpha_m y$, it is reasonable to assume that the coefficient A_m^* is identical to A_m. Hence, substituting Eq. (8) into Eq. (2), one obtains

$$\sum_{m=1}^{\infty} A_m\left(2+\beta_m+\frac{1}{\beta_m}\right)\cos\alpha_m y=\theta_b(y) \tag{9}$$

Consider a specific positive root of Eq. (7), α_n, and multiply both sides of Eq. (9) by $\cos\alpha_n y$ to integrate the equation with respect to y from $y=0$ to $y=1$.

Then,

$$\sum_{m=1}^{\infty} A_m\left(2+\beta_m+\frac{1}{\beta_m}\right)\int_0^1 \cos\alpha_m y\,\cos\alpha_n y\,\mathrm{d}y = \int_0^1 \theta_b(y)\cos\alpha_n y\,\mathrm{d}y \tag{10}$$

By combining the relations from Eq. (7) for α_m and α_n, it can be shown that $\cos\alpha_m y$ and $\cos\alpha_n y$ are orthogonal if the absolute values of these two roots are not the same. Hence,

$$\int_0^1 \cos\alpha_m y\,\cos\alpha_n y\,\mathrm{d}y=0 \qquad \text{for} \qquad |\alpha_m|\neq|\alpha_n| \tag{11}$$

Then, the coefficient A_m can be determined as

$$A_m=2B_y\Big/\left[\left(2+\beta_m+\frac{1}{\beta_m}\right)(B_y+\sin^2\alpha_m)\right] \times\int_0^1 \theta_b(y)\cos\alpha_m y\,\mathrm{d}y \tag{12}$$

The nondimensional temperature distribution at the base θ_b should be specified as one of the boundary conditions. However, this specified distribution should also satisfy Eqs. (4) and (5) to be consistent with the condition at the corners. If a second-order polynomial is assumed for this distribution, it can be shown that

$$\theta_b=(1+\gamma-\gamma y^2)\theta_s \tag{13}$$

where $\theta_s\equiv(T_s-T_f)/T_f$, $\gamma\equiv B_y/2$, and T_s is the edge temperature ($y=1$) at the base ($x=0$). Substituting Eq. (13) into Eq. (12) yields

$$A_m=\frac{2B_y^2\theta_s\sin\alpha_m}{\alpha_m^3(2+\beta_m+\beta_m^{-1})(B_y+\sin^2\alpha_m)} \tag{14}$$

The total heat transfer rate $\dot{Q}$ can then be determined from either one of the following two relations:

$$\frac{\dot{Q}}{T_f k_x}=-\frac{2b}{L}\int_0^1\left.\frac{\partial\theta}{\partial x}\right|_{x=0}\mathrm{d}y \tag{15}$$

$$\frac{\dot{Q}}{T_f k_x}=-\frac{2b}{L}\int_0^1\left.\frac{\partial\theta}{\partial x}\right|_{x=1}\mathrm{d}y-\frac{2Lk_x}{bk_y}\int_0^1\left.\frac{\partial\theta}{\partial y}\right|_{y=1}\mathrm{d}x \tag{16}$$

Substituting Eq. (8) into Eqs. (15) or (16), one obtains

$$\frac{\dot{Q}}{T_f k_x}=\sum_{m=1}^{\infty}\frac{2b}{L}A_m p(\beta_m-\beta_m^{-1})\sin\alpha_m \tag{17}$$

The fact that both Eqs. (15) and (16) reduce to the same relation proves the validity of the solution given in Eq. (8). If the base temperature distribution in Eq. (13) is assumed, the nondimensional heat transfer rate becomes

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \sum_{m=1}^{\infty} \frac{4bpB_y^2(\beta_m - \beta_m^{-1})}{L\alpha_m^3 (B_y + \sin^2\alpha_m)(2+\beta_m+\beta_m^{-1})} \tag{18}$$

and the nondimensional fin tip temperature θ_t at $x=1$ and $y=1$ can be shown as

$$\frac{\theta_t}{\theta_s} = \sum_{m=1}^{\infty} \left\{4pB_y \sin^2\alpha_m \left[\frac{e^{p\alpha_m}}{(p\alpha_m - B_x)} + \frac{e^{-p\alpha_m}}{(p\alpha_m + B_x)}\right]\right\}$$
$$\div [\alpha_m (B_y + \sin^2\alpha_m)(2+\beta_m+\beta_m^{-1})] \tag{19}$$

Formulation Using the Variational Principle for the Problem with Internal Heat Generation

The governing equation of heat conduction becomes nonhomogeneous if the internal heat generation is considered. Application of the variational principle is a practical means of obtaining an approximate solution of this problem, since it is a very difficult task to obtain the exact solution that satisfies the same convective boundary conditions. Considering the heat balance condition in a two-dimensional steady-state rectangular region with a rate of internal heat generation per unit volume $g(x,y)$, one can express the governing equation as follows:

$$\frac{\partial}{\partial x}\left(a_x \frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(a_y \frac{\partial\theta}{\partial y}\right) + g(x,y) = 0 \tag{20}$$

The coordinates x and y are normalized with respect to the length and width of the region, respectively. Assume that the temperature distribution and the heat sources function are symmetric about the X axis, as in the case without heat generation (Fig. 1). The boundary conditions expressed in Eqs. (2-5) can then be applied to this case.

Using the weighted-residual version of the variational principle, one can convert Eq. (20) into the following form:

$$-\int_0^1\int_0^1 \left[\frac{\partial}{\partial x}\left(a_x \frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(a_y \frac{\partial\theta}{\partial y}\right) + g(x,y)\right]\delta\theta \mathrm{d}x\, \mathrm{d}y = 0 \tag{21}$$

where $\delta\theta$ is the variation of the dependent variable θ.

Utilizing the boundary conditions in Eqs. (2-5), one can manipulate Eq. (21) into the following form:

$$\delta I = 0 \tag{22}$$

where

$$I \equiv \frac{1}{2}\int_0^1\int_0^1 \left[a_x\left(\frac{\partial\theta}{\partial x}\right)^2 + a_y\left(\frac{\partial\theta}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y - \int_0^1\int_0^1 g(x,y)\theta \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_0^1 C_x(\theta^2)_{x=1}\mathrm{d}y + \frac{1}{2}\int_0^1 C_y(\theta^2)_{y=1}\mathrm{d}x \tag{23}$$

Equation (22) implies that an admissible function that satisfies the boundary condition in Eqs. (2-5) can be an approximate solution of Eq. (20) if it produces a stationary value of the functional I.

The nondimensional total heat transfer rate can be evaluated by either one of the following relations:

$$\frac{\dot{Q}}{T_f k_x} = -\frac{2b}{L}\int_0^1 \frac{\partial\theta}{\partial x}\bigg|_{x=1}\mathrm{d}y - \frac{2Lk_y}{bk_x}\int_0^1 \frac{\partial\theta}{\partial y}\bigg|_{y=1}\mathrm{d}x \tag{24}$$

$$\frac{\dot{Q}}{T_f k_x} = \frac{2b}{L}\left(\int_0^1\int_0^1 \frac{g}{a_x}\mathrm{d}x\mathrm{d}y - \int_0^1 \frac{\partial\theta}{\partial x}\bigg|_{x=0}\mathrm{d}y\right) \tag{25}$$

In general, the total heat transfer rate calculated from these two equations will not be consistent if the trial function for the temperature distribution does not satisfy the governing equation, Eq. (20). Under this circumstance, the solution does not correspond completely to a physical phenomenon even though it should converge to the exact solution mathematically. To overcome this shortcoming, the following relation can be imposed as an additional boundary condition to determine the trial function:

$$\int_0^1 \frac{\partial\theta}{\partial x}\bigg|_{x=0}\mathrm{d}y - \int_0^1\int_0^1 \frac{g}{a_x}\mathrm{d}x\mathrm{d}y = \int_0^1 \frac{\partial\theta}{\partial x}\bigg|_{x=1}\mathrm{d}y + \frac{a_y}{a_x}\int_0^1 \frac{\partial\theta}{\partial y}\bigg|_{y=1}\mathrm{d}x \tag{26}$$

Table 1 Comparisons of the nondimensional total heat transfer rates ($\dot{Q}/k_x T_f \theta_s$) for the case without internal heat generation ($h_x/h_y = 1$, $k_x/k_y = 1$) (Results in A and B are calculated from Eqs. (25) and (24), respectively)

			ES	RRM1		RRM2		RRM/HB	RSM		RSM/HB	FDM	HBIM	MFT	
B_y	B_x	L/b	A&B	A	B	A	B	A&B	A	B	A&B	B	A&B	A	B
	0.001	0.1	0.0221	0.0221	0.0221	0.0221	0.0221	0.0221	0.0221	0.0221	0.0221	0.0362	0.0221	0.0220	0.00198
	0.005	0.5	0.0299	0.0299	0.0299	0.0299	0.0299	0.0299	0.0299	0.0299	0.0299	0.0483	0.0299	0.0299	0.00984
0.01	0.01	1	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.0396	0.01954
	0.02	2	0.0584	0.0582	0.0584	0.0584	0.0584	0.0854	0.0585	0.0584	0.0584	0.0387	0.0584	0.0583	0.03833
	0.1	10	0.1603	0.1498	0.1602	0.1586	0.1603	0.1576	0.1645	0.1603	0.1605	0.0962	0.1603	0.1601	0.1402
	0.01	0.1	0.2244	0.2244	0.2244	0.2244	0.2244	0.2244	0.2244	0.2244	0.2244	0.3682	0.2244	0.2241	0.0178
	0.05	0.5	0.2914	0.2907	0.2914	0.2914	0.2914	0.2914	0.2917	0.2914	0.2914	0.4550	0.2914	0.2910	0.0847
0.1	0.10	1	0.3651	0.3617	0.3650	0.3649	0.3651	0.3648	0.3664	0.3651	0.3651	0.3651	0.3651	0.3646	0.1584
	0.20	2	0.4771	0.4622	0.4768	0.4757	0.4772	0.4745	0.4835	0.4771	0.4773	0.3481	0.4772	0.4766	0.2705
	1	10	0.6415	0.4193	0.6485	0.5792	0.6473	0.4823	1.0727	0.6361	0.6672	0.6211	0.6416	0.6400	0.4350
	0.1	0.1	2.606	2.602	2.606	2.604	2.606	2.606	2.605	2.606	2.606	4.215	2.606	2.603	−0.0603
	0.5	0.5	2.450	2.400	2.452	2.444	2.455	2.442	2.470	2.454	2.455	3.207	2.453	2.446	−0.215
1	1	1	2.362	2.214	2.356	2.347	2.374	2.311	2.453	2.370	2.372	2.366	2.369	2.357	−0.302
	2	2	2.311	1.921	2.266	2.246	2.337	2.087	2.684	2.307	2.331	2.154	2.320	2.304	−0.351
	10	10	2.230	0.798	2.585	1.397	2.822	0.922	−2.207	2.060	2.667	3.049	2.309	2.280	−0.345
	1	0.1	44.49	44.27	44.52	44.37	44.53	44.46	44.44	44.53	44.53	59.76	44.50	44.37	−42.11
	5	0.5	18.09	17.06	18.09	17.82	18.43	17.65	18.63	18.29	18.30	19.19	18.23	17.97	−68.23
10	10	1	13.82	11.88	13.65	13.38	14.58	12.70	15.48	14.14	14.18	13.93	14.13	13.70	−72.14
	20	2	12.78	9.07	12.29	11.64	14.25	10.01	18.39	13.08	13.39	13.29	13.21	12.64	−72.48
	100	10	12.24	2.91	19.15	5.41	21.84	3.34	−4.09	11.58	17.51	23.88	13.17	12.45	−66.96

The Rayleigh-Schmidt Method

The conventional Rayleigh-Ritz method in which an admissible function is assumed in polynomial form can be utilized to optimize the functional in Eq. (23). Solutions obtained by this method for the case of constant rate of heat generation are listed in Appendix A. Appendix B gives the solution of the same case with the heat balance constraint of Eq. (26) being imposed as an additional boundary condition.

The polynomial form of trial functions used in the Rayleigh-Ritz method can be replaced by a power law as follows to improve its accuracy:

$$\theta/\theta_s = \theta_b(y)(1 + C_1 x + C_2 x^m) \tag{27}$$

where m is an undetermined constant. The function satisfies all boundary conditions except that of Eq. (3). Imposing this condition, one can express the undetermined coefficient C_1 in terms of C_2 and m. Then, the functional I can be determined in terms of these two parameters by substituting the admissible function into Eq. (23). Two transcendental simultaneous equations in C_2 and m can be obtained by minimizing the functional I with respect to C_2 and with respect to m as follows:

$$\partial I/\partial m = 0$$

$$\partial I/\partial C_2 = 0 \tag{28}$$

Newton's method can be used to solve the two unknowns, and the nondimensional heat transfer rate can be determined from either Eq. (24) or (25) as follows:

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{2L\gamma k_y}{bk_x}\left\{1 - \frac{1}{(B_x+1)} - C_2(m-1)\left[\frac{1}{(m+1)} + \frac{1}{(B_x+1)}\right]\right\} + \frac{2b}{3L}(B_y+3)\left[1 - \frac{1}{(B_x+1)}\right] \times [C_2(m-1) - 1] \tag{29}$$

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{2b}{L}\left\{\frac{g}{a_x\theta_s} + \frac{(B_y+3)}{3(B_x+1)}[(B_x+m)C_2 + B_x]\right\} \tag{30}$$

To eliminate the inconsistency between the computed heat transfer rates, Eq. (26) can be taken as an additional boundary condition to be satisfied by the trial function in Eq. (27). Then, one can eliminate the coefficient C_2 and obtain the transcendental relation of the undetermined exponent m as next shown by minimizing the functional I with respect to m in the form of the first of Eqs. (28). Thus,

$$\left(\frac{4a_y\gamma^2}{3} + C_y\right)\left\{b_5\left(\frac{1}{2} + \frac{C_1}{3} + \frac{C_2}{m+2}\right) + b_6\left(\frac{C_2}{2m+1} + \frac{C_1}{m+2} + \frac{1}{m+1}\right) - C_2\left[\frac{C_2}{(2m+1)^2} + \frac{C_1}{(m+2)^2} + \frac{1}{(m+1)^2}\right]\right\} + b_5 b_7[(C_1+C_2)(a_x+C_x) + C_x] + b_6 b_7\left[a_x\left(\frac{m^2 C_2}{2m-1} + C_1\right) + C_x(1 + C_1 + C_2)\right] + \frac{a_x b_7 m(m-1)C_2^2}{(2m-1)^2} - \frac{g}{3\theta_s}(3+B_y) \times \left[\frac{b_5}{2} + \frac{b_6}{m+1} - \frac{C_2}{(m+1)^2}\right] = 0 \tag{31}$$

where

$$C_2 \equiv b_3/b_4$$

$$b_3 \equiv \gamma p^2\left[1 + \frac{1}{(B_x+1)}\right] - \frac{g}{a_x\theta_s}$$

$$b_4 \equiv \beta p^2\left(\frac{m-1}{m+1}\right) + \frac{m}{3}(3+B_y) + \frac{\gamma p^2(m-1)}{(B_x+1)}$$

$$b_5 \equiv \frac{-1}{(B_x+1)}[C_2 + b_6(m-1)] - b_6$$

$$b_6 \equiv -\frac{C_2}{b_4}\left\{\gamma p^2\left[\frac{2}{(m+1)^2} + \frac{1}{B_x+1}\right] + \frac{(3+B_y)}{3}\right\}$$

$$b_7 \equiv 1 - \frac{2}{3}B_y + \frac{2}{15}B_y^2$$

$$C_1 \equiv \frac{-1}{B_x+1}[B_x + C_2(B_x+m)]$$

Table 2 Comparisons of the nondimensional tip temperatures (θ/θ_s) at $x = y = 1$ for the case without internal heat generation ($h_x/h_y = 1$, $k_x/k_y = 1$)

B_y	B_x	L/b	ES	RRM1	RRM2	RRM/HB	RSM	RSM/HB	FDM	HBIM	MFT
	0.001	0.1	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990
	0.005	0.5	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938	1.1855	0.9938	0.9938
0.01	0.01	1	0.9852	0.9852	0.9852	0.9852	0.9852	0.9852	0.9852	0.9852	0.9852
	0.02	2	0.9614	0.9614	0.9614	0.9613	0.9614	0.9614	0.5539	0.9614	0.9614
	0.1	10	0.6029	0.5969	0.6036	0.5806	0.6024	0.6027	0.0411	0.6029	0.6030
	0.01	0.1	0.9896	0.9896	0.9896	0.9896	0.9896	0.9896	0.9897	0.9896	0.9896
	0.05	0.5	0.9412	0.9413	0.9413	0.9412	0.9413	0.9413	1.1007	0.9413	0.9413
0.1	0.1	1	0.8691	0.8690	0.8693	0.8694	0.8692	0.8693	0.8691	0.8693	0.8692
	0.2	2	0.7076	0.7058	0.7078	0.7011	0.7075	0.7076	0.4542	0.7077	0.7077
	1	10	0.0674	−0.0097	0.0833	−09838	0.0413	0.0407	0.0129	0.0674	0.0682
	0.1	0.1	0.9051	0.9059	0.9059	0.9059	0.9059	0.9059	0.9050	0.9059	0.9052
	0.5	0.5	0.6162	0.6200	0.6213	0.6173	0.6120	0.6211	0.6570	0.6207	0.6166
1	1	1	0.3917	0.3899	0.3968	0.3797	0.3950	0.3952	0.3922	0.3954	0.3925
	2	2	0.1636	0.1398	0.1685	0.1142	0.1579	0.1581	0.1433	0.1646	0.1653
	10	10	1.67×10^{-4}	−0.0827	0.0529	−0.1130	−0.0352	−0.0427	2.20×10^{-4}	1.61×10^{-4}	8.27×10^{-3}
	1	0.1	0.4890	0.4964	0.4966	0.4958	0.4965	0.4965	0.4866	0.4962	0.4898
	5	0.5	0.1369	0.1451	0.1491	0.1409	0.1475	0.1475	0.1362	0.1471	0.1410
10	10	1	0.0554	0.0524	0.0631	0.0454	0.0582	0.0580	0.0558	0.0600	0.0635
	20	2	0.0126	−0.0042	0.0185	−0.0132	0.0065	0.0059	0.0133	0.0127	0.0288
	100	10	1.36×10^{-7}	−0.0133	0.0140	−0.0175	−0.0051	−0.0069	3.00×10^{-6}	6.66×10^{-8}	0.0806

Table 3 Comparisons of the nondimensional total heat transfer rates ($\dot{Q}/k_x T_f \theta_s$) for the case with internal heat generation ($h_x/h_y = 1$, $k_x/k_y = 1$, $g/a_x\theta_s = 5$) (Results in A and B are calculated from Eqs. (25) and (24), respectively)

			RRM1		*RRM2*		*RRM/HB*	*RSM*		*RSM/HB*	*B*	*HBIM*	*MFT*	
B_y	B_x	L/b	*A*	*B*	*A*	*B*	*A&B*	*A*	*B*	*A&B*	*B*	*A&B*	*A*	*B*
	0.001	0.1	0.0762	0.0753	11.93	0.0773	0.0753	0.0754	0.0753	0.0753	0.1246	1.00×10^6	0.1563	0.0553
	0.005	0.5	0.0995	0.0961	2.465	0.0981	0.0962	0.0954	0.0961	0.0961	0.1601	7940	0.1123	0.0761
0.01	0.01	1	0.128	0.122	1.302	0.124	0.122	0.120	0.122	0.122	0.122	979	0.123	0.1017
	0.02	2	0.184	0.171	0.754	0.173	0.171	0.168	0.171	0.171	0.101	118	0.175	0.1510
	0.1	10	0.453	0.420	0.491	0.420	0.428	0.410	0.419	0.419	0.178	0.964	0.420	0.3993
	0.01	0.1	0.778	0.751	12.38	0.771	0.751	0.751	0.751	0.751	1.239	1.00×10^6	0.833	0.545
	0.05	0.5	0.951	0.917	3.215	0.936	0.918	0.915	0.917	0.917	1.475	740.1	0.935	0.712
0.1	0.1	1	1.149	1.094	2.203	1.111	1.098	1.084	1.093	1.093	1.092	81.94	1.102	0.888
	0.2	2	1.421	1.342	1.821	1.354	1.355	1.325	1.341	1.340	0.883	8.653	1.345	1.135
	1	10	1.194	1.331	1.091	1.331	1.231	−0.506	1.315	1.332	1.047	1.328	1.327	1.121
	0.1	0.1	8.59	7.31	17.43	7.46	7.37	8.53	7.31	7.32	11.78	10880	7.51	4.76
	0.5	0.5	6.78	6.45	8.21	6.56	6.51	6.67	6.45	6.45	8.67	39.60	6.58	3.90
1	1	1	5.92	5.76	6.29	5.81	5.82	5.90	5.76	5.77	5.82	5.80	5.84	3.17
	2	2	4.81	4.90	4.62	4.86	4.86	5.05	4.94	4.99	4.35	4.69	4.97	2.31
	10	10	1.74	3.42	1.56	3.35	1.86	−0.75	2.87	3.48	3.81	3.19	3.17	0.540
	1	0.1	77.54	67.19	62.01	64.97	69.61	77.62	67.20	67.28	92.70	69.83	70.44	−16.13
	5	0.5	27.33	26.24	23.69	24.59	26.73	28.27	26.52	26.73	29.29	6.88	27.50	−58.71
10	10	1	17.98	18.66	15.38	17.05	18.31	16.80	18.41	18.55	19.67	17.32	19.48	−66.37
	20	2	12.83	15.46	10.81	13.92	13.62	24.70	16.04	16.41	16.67	16.57	16.20	−68.92
	100	10	3.89	20.34	4.44	20.85	4.31	−3.02	12.37	18.34	24.73	14.10	13.39	−66.02

Table 4 Comparisons of the nondimensional tip temperatures (θ/θ_s) at $x = y = 1$ for the case with internal heat generation ($h_x/h_y = 1$, $k_x/k_y = 1$, $g/a_x\theta_s = 5$)

B_y	B_x	L/b	RRM1	RRM2	RRM/HB	RSM	RSM/HB	FDM	HBIM	MFT
	0.001	0.1	3.488	3.587	3.488	3.488	3.488	3.496	-4.99×10^5	3.495
	0.005	0.5	3.471	3.569	3.471	3.471	3.471	4.143	−1975	3.473
0.01	0.01	1	3.443	3.540	3.444	3.442	3.442	3.443	−485	3.443
	0.02	2	3.367	3.460	3.372	3.365	3.365	1.939	−115	3.365
	0.1	10	2.251	2.280	2.303	2.234	2.234	0.152	−0.953	2.234
	0.01	0.1	3.384	3.477	3.384	3.384	3.384	3.456	−4896	3.449
	0.05	0.5	3.224	3.314	3.227	3.223	3.223	3.795	−176	3.248
0.1	0.1	1	2.992	3.075	3.002	2.989	2.989	2.999	−36.8	2.998
	0.2	2	2.483	2.544	2.509	2.475	2.475	1.591	−5.07	2.478
	1	10	0.354	0.294	0.304	0.371	0.370	0.076	0.412	0.413
	0.1	0.1	2.585	2.639	2.605	2.585	2.586	3.105	−406	3.044
	0.5	0.5	1.791	1.832	1.809	1.789	1.790	2.049	−4.600	1.919
1	1	1	1.176	1.195	1.188	1.175	1.176	1.219	1.184	1.213
	2	2	0.536	0.519	0.529	0.553	0.557	0.494	0.884	0.564
	10	10	−0.0658	−0.1066	−0.0941	−0.0231	−0.0298	0.0145	0.0475	0.0316
	1	0.1	0.753	0.728	0.781	0.753	0.754	1.507	0.783	1.321
	5	0.5	0.225	0.206	0.230	0.229	0.230	0.302	1.530	0.289
10	10	1	0.0899	0.0713	0.0874	0.0885	0.0893	0.1175	0.4871	0.1190
	20	2	0.0086	−0.0093	0.0014	0.0175	0.0169	0.0341	0.1307	0.0467
	100	10	−0.0130	−0.0071	−0.0173	−0.0050	−0.0067	0.0013	0.0050	0.0814

The unknown m can then be determined by the Bolzano method[12] (or the method of halving the interval) and the total heat transfer rate can be evaluated by either Eq. (29) or (30).

Numerical Results

All of the techniques discussed previously are employed in the numerical computations for the two-dimensional heat transfer problem. In addition, the FDM, and MFT, and the HBIM are also included for comparison. All of these methods are suitable for the cases with and without internal heat generation. Formulations for the HBIM and the MFT are listed in Appendices C and D, respectively. The nondimensionalized domain ($0 \le x \le 1$ and $0 \le y \le 1$) of one-half of the region is divided into 36 equally spaced modes for the FDM.

Three different trial functions are employed for the RRM: a second-order polynomial function as shown in Eq. (A1) (RRM1); a third-order polynomial function listed in Eq. (A3) (RRM2); and the third-order polynomial function with an additional heat-balance boundary condition expressed in general form in Eq. (26) (RRM/HB). Two approaches are adopted to apply the RSM: one without the additional heat-balance condition as described in Eqs. (29) and (30) (RSM), and one with the additional condition as shown in Eq. (31) (RSM/HB). Results calculated from approximate techniques for cases without internal heat generation are compared with the exact solution (ES) in Table 1.

Table 2 summarizes the comparisons of tip temperatures at $x = 1$ and $y = 1$ predicted from these techniques under the same conditions. The results obtained from all methods agree reasonably well with each other with one exception, namely those calculated from the finite-difference methods for some cases. Discrepancies of tip temperature occur for some cases when the temperatures become small. Since this is the temperature at one point only, it can happen as either the results of roundoff error or poor fittings of the trial functions at that point.

The method of Fourier transform (MFT) predicts reasonable heat transfer rates by utilizing Eq. (25) for the computations. The data presented were calculated to converge by taking 250 terms in x and 250 terms in y. Also, the

Table 5 Comparisons of the nondimensional total heat transfer rates ($\dot{Q}/k_xT_x\theta_s$) with variations of the directional convective heat transfer coefficents for the case without internal heat generation ($L/b=1$, $k_x/k_y=1$) (Results in A and B are calculated from Eqs. (25) and (24), respectively)

			ES	RRM1		RRM2		RRM/HB	RSM		RSM/HB	FDM	HBIM	MFT	
B_y	B_x	h_x/h_y	A & B	A	B	A	B	A & B	A	B	A & B	B	A & B	A	B
	0.001	0.1	0.02192	0.02190	0.02192	0.02192	0.02192	0.02192	0.02191	0.02192	0.02192	0.02192	0.02192	0.02190	0.01991
	0.005	0.05	0.02982	0.02979	0.02982	0.02982	0.02982	0.02982	0.02983	0.02982	0.02982	0.02982	0.02982	0.02979	0.01979
0.01	0.01	1	0.03961	0.03957	0.03961	0.03960	0.03961	0.03961	0.03962	0.03961	0.03961	0.03961	0.03961	0.03956	0.01954
	0.02	2	0.05889	0.05883	0.05890	0.05889	0.05889	0.05889	0.05892	0.05889	0.05889	0.05889	0.05889	0.05883	0.01876
	0.1	10	0.2006	0.2004	0.2006	0.2006	0.2006	0.2006	0.2007	0.2006	0.2006	0.2006	0.2006	0.2003	-4.42×10^{-5}
	0.001	0.1	0.2124	0.2109	0.2124	0.2122	0.2124	0.2124	0.2128	0.2124	0.2124	0.2124	0.2124	0.2122	0.1917
	0.05	0.5	0.2834	0.2810	0.2834	0.2832	0.2834	0.2833	0.2842	0.2834	0.2834	0.2834	0.2834	0.2831	0.1800
0.1	0.1	1	0.3651	0.3617	0.3650	0.3649	0.3651	0.3648	0.3664	0.3651	0.3651	0.3651	0.3651	0.3646	0.1584
	0.2	2	0.5086	0.5035	0.5084	0.5083	0.5086	0.5080	0.5108	0.5086	0.5086	0.5085	0.5086	0.5079	0.0953
	1	10	1.148	1.136	1.146	1.148	1.148	1.144	1.156	1.148	1.148	1.148	1.148	1.145	−0.917
	0.1	0.1	1.731	1.638	1.744	1.713	1.745	1.722	1.759	1.745	1.746	1.734	1.741	1.729	1.464
	0.5	0.5	2.093	1.966	2.098	2.076	2.105	2.062	2.153	2.104	2.106	2.096	2.100	2.089	0.760
1	1	1	2.362	2.214	2.356	2.347	2.374	2.311	2.453	2.370	2.372	2.366	2.369	2.357	−0.302
	2	2	2.652	2.484	2.626	2.637	2.663	2.574	2.781	2.651	2.655	2.657	2.657	2.644	−2.673
	10	10	3.113	2.932	3.018	3.101	3.124	2.979	3.302	3.077	3.083	3.120	3.116	3.077	−23.3
	1	0.1	12.48	10.61	13.22	11.98	13.38	12.01	13.40	13.32	13.32	12.54	12.91	12.54	3.82
	5	0.5	13.53	11.56	13.64	13.07	14.31	12.57	15.08	14.02	14.05	13.63	13.86	13.46	−29.6
10	10	1	13.82	11.88	13.65	13.38	14.58	12.70	15.48	14.14	14.18	13.93	14.13	13.70	−72.1
	20	2	14.00	12.10	13.62	13.57	14.75	12.78	15.67	14.19	14.23	14.11	14.30	13.77	−157
	100	10	14.17	12.31	13.56	13.75	14.91	12.84	15.80	14.22	14.26	14.28	14.45	13.06	−783

Table 6 Comparisons of the nondimensional total heat transfer rates ($\dot{Q}/k_xT_f\theta_s$) with variations of the directional thermal conductivities for the case without internal heat generation ($h_x/h_y=1$, $L/b=1$) (Results in A and B are calculated frrm Eqs. (25) and (24), respectively.)

			ES	RRM1		RRM2		RRM/HB	RSM		RSM/HB	FDM	HBIM	MFT	
B_y	B_x	k_x/k_y	A & B	A	B	A	B	A & B	A	B	A & B	B	A & B	A	B
	0.1	0.1	0.3595	0.3561	0.3594	0.3593	0.3595	0.3592	0.3608	0.3595	0.3595	0.1708	0.3595	0.3591	0.1588
	0.02	0.5	0.07831	0.07817	0.07831	0.07831	0.07831	0.07831	0.07837	0.07831	0.07831	0.06162	0.07831	0.07823	0.03818
0.01	0.01	1	0.03961	0.03957	0.03961	0.03960	0.03961	0.03961	0.03962	0.03961	0.03961	0.03961	0.03961	0.03956	0.01954
	0.005	2	0.01992	0.01991	0.01992	0.01992	0.01992	0.01992	0.01992	0.01992	0.01992	0.02473	0.01992	0.01990	0.00988
	0.001	10	0.00400	0.00400	0.00400	0.00400	0.00400	0.00400	0.00400	0.00400	0.00400	0.00690	0.00400	0.00400	0.00200
	1	0.1	2.038	1.885	2.020	2.024	2.039	1.974	2.141	2.034	2.038	1.404	2.038	2.033	−0.0274
	0.2	0.5	0.6644	0.6526	0.6641	0.6637	0.6645	0.6628	0.6696	0.6644	0.6645	0.5476	0.6644	0.6636	0.2511
0.1	0.1	1	0.3651	0.3617	0.3650	0.3649	0.3651	0.3648	0.3664	0.3651	0.3651	0.3651	0.3651	0.3646	0.1584
	0.05	2	0.1923	0.1914	0.1923	0.1922	0.1923	0.1923	0.1926	0.1923	0.1923	0.2356	0.1923	0.1921	0.0889
	0.01	10	0.0402	0.0402	0.0402	0.0402	0.0402	0.0402	0.0402	0.0402	0.0402	0.0691	0.0402	0.0402	0.0195
	10	0.1	7.31	5.34	6.65	6.82	7.54	5.77	10.48	7.03	7.27	7.33	7.34	7.26	−19.1
	2	0.5	3.39	3.03	3.34	3.35	3.42	3.21	3.69	3.39	3.41	3.21	3.40	3.38	−1.93
1	1	1	2.362	2.214	2.356	2.347	2.374	2.311	2.453	2.370	2.372	2.366	2.369	2.357	−0.302
	0.5	2	1.535	1.480	1.537	1.528	1.541	1.525	1.559	1.540	1.541	1.747	1.539	1.533	0.202
	0.1	10	0.420	0.416	0.421	0.418	0.421	0.421	0.420	0.421	0.421	0.697	0.421	0.420	0.154
	100	0.1	40.2	23.1	38.3	33.2	47.7	25.7	104.3	40.3	43.2	45.2	41.6	39.0	−757
	29	0.5	18.5	14.7	17.9	17.5	20.0	15.9	22.6	18.9	19.0	18.8	19.0	18.2	−152
10	10	1	13.82	11.88	13.65	13.38	14.58	12.70	15.48	14.14	14.18	13.93	14.13	13.70	−72.1
	5	2	10.67	9.67	10.70	10.43	11.04	10.23	11.31	10.90	10.91	11.02	10.83	10.60	−32.4
	1	10	5.42	5.23	5.48	5.34	5.49	5.41	5.41	5.49	5.49	7.10	5.46	5.41	−3.24

heat transfer rates calculated by Eq. (24) for the MFT do not yield satisfactory results. Apparently, the temperature gradients along the edge $y=1$ in the x direction do not converge to the true values even though the tip temperatures agree reasonably well with those predicted by different techniques. These discrepancies indicate clearly the disadvantages of the MFT in heat transfer applications.

Imposing the heat balance boundary condition for both the conventional Rayleigh-Ritz method and the revised Rayleigh method (RSM) not only eliminates the inconsistency between results predicted from Eqs. (24) and (25), but also improves the accuracy of heat transfer rates as they are computed from Eq. (25). However, it does not improve the accuracy in the computation of tip temperature for most cases and, in general, its results in heat transfer rate are also less accurate than those obtained from Eq. (24) using the same method but excluding this condition of heat balance. In general, the RSM yields the best results if the heat transfer rate is calculated from Eq. (24). However, the differences between results predicted from various techniques are not very significant, in general, if no internal heat is generated in the region.

Comparisons of the results for the cases of uniform directional conductivities and heat transfer coefficients with internal heat generation are presented in Table 3. Table 4 summarizes the tip temperature calculated from various approximate techniques for these cases. The nondimensional heat generation parameter ($g/\theta_s a_x$) is selected to be 5 for all cases. All results predicted except those calculated from the HBIM agree reasonably well with one another. In all cases, the HBIM is incapable of predicting reasonable results, and hence, the technique becomes unreliable for the case with internal heat generation. Again, the total heat transfer rates predicted by the MFT are unsatisfactory by employing Eq. (24) instead of Eq. (25). Hence, limitations of the MFT do exist in heat transfer applications, and caution should be exercised in applying this technique to engineering design.

Comparisons of heat transfer rates predicted by different techniques for the cases with variations of the ratios of the directional convective heat transfer coefficients and directional thermal conductivity ratios are presented in Tables 5 and 6, respectively. Again, reasonably good agreements are observed among different techniques except those calculated by Eq. (24) in the MFT. Since these cases do not include internal heat generation, the problems of the HBIM do not exist in these results.

Conclusions

Various approximate solutions of a two-dimensional steady conduction problem with convective boundary conditions have been successfully applied, and results obtained agree fairly well among the various approximate techniques. In addition, the exact solution for the case without internal heat generation has been derived successfully, and results obtained from it agree fairly well with those obtained from existing approximate techniques. It has been proven from the results presented previously that the existing HBIM becomes unreliable for the case with internal heat generation, and the MFT has very serious limitations for the calculations of heat transfer rates in both the cases with and without internal heat generation.

The Rayleigh-Ritz method is a very viable approximate technique to solve two-dimensional heat transfer problems. For many cases in steady conduction without internal heat generation, results obtained by it have better accuracy than those calculated from the fintite-difference method. Therefore, it is reasonable to expect that it has the same trend for the cases with internal heat generation. This speculation can be verified by comparing results with those predicted by the MFT. The general agreement between results obtained from both techniques for cases with and without internal heat generation is a good indication that they are reliable to predict results for the case with internal heat generation. However, it has been shown from the results that the RSM can also be a powerful technique in solving heat transfer problems. In most cases, it yields even better resuts than those from the RRM and FDM. Hence, it can be used with confidence for the two-dimensional heat transfer problem with internal heat generation.

The problem of the inconsistency in estimating heat transfer rates by integrating across different boundaries of the region in both RRM and the RSM can be eliminated by imposing the heat balance condition as an additional boundary condition for the trial functions. The loss in accuracy has been demonstrated to be very insignificant. Therefore, it is also reasonable to incorporate this scheme into both techniques in solving a heat tranfer problem.

Appendix A: Heat Transfer Rates and Temperature Distributions of the Conventional RRM

The total heat transfer rates evaluated from Eqs. (24) and (25) using the nth-order polynomial are

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{4L\gamma}{b}\left(\frac{k_y}{k_x}\right)\left\{1 + \sum_{k=2}^{n} C_k\left[\frac{1}{k+1} - \frac{(k+B_x)}{2(B_x+1)}\right]\right\} - \frac{2b}{3L}(2\gamma+3)\sum_{k=2}^{n} C_k\left[k - \frac{(k+B_x)}{(B_x+1)}\right] \tag{A1}$$

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{2b}{L}\left\{\frac{(B_y+3)}{3(B_x+1)}\left[B_x + \sum_{k=1}^{n}(B_x+k)C_k\right] + \frac{g}{a_x\theta_s}\right\} \tag{A2}$$

Case A: $n=2$

The temperature distribution is

$$\frac{\theta}{\theta_s} = (1+\gamma-\gamma y^2)\left\{1 - \frac{B_x}{(B_x+1)}x + C_2\left[x^2 - \left(\frac{B_x+2)}{B_x+1)}\right)x\right]\right\} \tag{A3}$$

where

$$C_2 = F_3/F_1$$

$$F_1 \equiv \frac{1}{3}a_2\left(\frac{1}{10} + \frac{a_1}{2} + a_1^2\right) + C_x a_3 a_1^2 + a_x a_3\left(\frac{1}{3} + a_1^2\right)$$

$$F_3 \equiv a_2\left(\frac{1}{12} + \frac{a_1}{4} - \frac{a_1^2}{3}\right) + C_x a_3 a_1^2 + a_x a_3 a_1(a_1-1) - \frac{a_4}{2}\left(\frac{1}{3} + a_1\right)$$

$$a_1 \equiv 1/(B_x+1); \qquad a_2 \equiv C_y + (a_y B_y^2/3)$$

$$a_3 \equiv 1 - \frac{2}{3}B_y + \frac{2}{15}B_y^2; \quad a_4 \equiv \frac{g}{\theta_s}\left(1 + \frac{B_y}{3}\right)$$

Case B: $n=3$

The temperature distribution is

$$\frac{\theta}{\theta_s} = (1+\gamma-\gamma y^2)\left\{1 - \frac{B_x x}{(B_x+1)} + C_2\left[x^2 - \left(\frac{B_x+2}{B_x+1}\right)x\right] + C_3\left[x^3 - \left(\frac{B_x+3}{B_x+1}\right)x\right]\right\} \tag{A4}$$

The coefficients C_2 and C_3 are zeros of the following simultaneous equations:

$$\begin{bmatrix} F_1 & F_2 \\ F_2 & F_4 \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_5 \end{Bmatrix} \tag{A5}$$

where

$$F_2 \equiv a_2[1/20 + (3/10)a_1 + (2/3)a_1^2] + a_x a_3[(1/2) + 2a_1^2] + 2C_x a_3 a_1^2$$

$$F_4 \equiv (4a_2/3)[(2/35) + (2/5)a_1 + a_1^2] + 4a_x a_3[(1/5) + a_1^2] + 4C_x a_3 a_1^2$$

$$F_5 \equiv (a_2/2)[(7/30) + (14/15)a_1 + (4/3)a_1^2] + 2a_x a_3 a_1(a_1-1) + 2C_x a_3 a_1^2 - a_4[(1/4) + a_1]$$

The quantities a_1, a_2, a_3, a_4, F_1, and F_3 are the same quantities as those defined in Case A.

Appendix B: Temperature Distribution and Heat Transfer Rate of the RRM with the Heat Balance Constraint

The temperature distribution is assumed as

$$\frac{\theta}{\theta_s} = (1+\gamma-\gamma y^2)[1 - e_1 x + e_2 x^2 + C_3(x^3 + e_3 x^2 - e_4 x)] \tag{B1}$$

where

$$e_1 \equiv a_1(e_2 - 1) + e_2 + 1; \qquad e_2 \equiv b_1/b_2$$

$$e_3 \equiv -[p^2\gamma(½ + 2a_1) + B_y + 3]/b_2$$

$$e_4 \equiv e_3(1+a_1) + 1 + 2a_1$$

Other symbols used are consistent with those defined in Appendix A. The coefficient C_3 is determined as follows:

$$C_3 = F_7/F_6 \tag{B2}$$

where

$$F_6 \equiv a_2[1/7 + (1/5)e_3^2 + (1/3)e_4^2 - (2/5)e_4 - (1/2)e_3e_4 + (1/3)e_3] + a_x a_3[(9/5) + (4/3)e_3^2 + e_4^2 + 3e_3 - 2e_3e_4 - 2e_4] + C_x a_3(1 + e_3^2 + e_4^2 - 2e_4 - 2e_3e_4 + 2e_3)$$

$$F_7 \equiv a_2[-1/4 + (1/5)e_1 - (1/6)e_2 - (1/3)e_3 + (1/2)e_4 + (1/4)e_1e_3 - (1/3)e_1e_4 - (1/5)e_2e_3 + (1/4)e_2e_4] + a_x a_3[e_1 + e_1e_3 - e_1e_4 - (3/2)e_2 - (4/3)e_2e_3 + e_2e_4] + C_x a_3(-1 + e_1 - e_2 - e_3 + e_4 + e_1e_3 - e_1e_4 - e_2e_3 + e_2e_4) + a_4[1/4 + (1/3)e_3 - (1/2)e_4]$$

The total heat transfer rate is

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{2b}{L}\left[\left(1 + \frac{B_y}{3}\right)(e_1 + C_3 e_4) + \frac{g}{a_x \theta_s}\right] \tag{B3}$$

Appendix C: Temperature Distributions and Heat Transfer for the Case of Uniform Heat Generation Using the HBIM

The nondimensional temperature profile is assumed to be

$$\theta = \theta_0(x)\,(1 + \gamma - \gamma y^2) \tag{C1}$$

where θ_0 is the unknown edge temperature at $y = 1$. Integrating Eq. (20) with respect to y from $y = 0$ to $y = 1$, one obtains the following second-order differential equation in θ_0 for uniform heat generation:

$$\left(1 + \frac{B_y}{3}\right)\frac{d^2\theta_0}{dx^2} - \frac{C_y}{a_x}\theta_0 + \frac{g}{a_x} = 0 \tag{C2}$$

The solution of Eq. (C2) that satisfies the boundary conditions is

$$\frac{\theta_0}{\theta_s} = A \sinh\alpha x + (1-\beta)\cosh\alpha x + \beta \tag{C3}$$

where

$$\alpha \equiv \sqrt{\frac{3C_y}{a_x(3+B_y)}}; \qquad \beta \equiv \frac{g}{\theta_s C_y}$$

$$A \equiv \frac{1}{(\alpha\cosh\alpha + B_x\sinh\alpha)} \times [(\beta - 1)(B_x\cosh\alpha + \alpha\sinh\alpha) - \beta] \tag{C4}$$

The total heat transfer rate evaluted from Eq. (24) becomes

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{2b}{3L}(3 + B_y)\alpha[(\beta - 1)\sinh\alpha + A\cosh\alpha] + \frac{2Lk_y B_y}{bk_x\alpha}[A\cosh\alpha + (1-\beta)\sinh\alpha - A + \alpha\beta] \tag{C5}$$

If Eq. (C1) is substituted into Eq. (25), the total heat transfer rate can be determined as follows:

$$\frac{\dot{Q}}{k_x T_f \theta_s} = \frac{2b}{L}\left[\frac{g}{\theta_s a_x} - \frac{A\alpha}{3}(3 + B_y)\right] \tag{C6}$$

Appendix D: Temperature Distribution and Heat Transfer for the Case of Uniform Heat Generation Using the MFT

The governing equation, Eq. (20), and boundary condition, Eqs. (2-5), can be transfered into the following forms for constant a_x and a_y:

$$\frac{\partial}{\partial x}\left(a_x\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(a_y\frac{\partial\phi}{\partial y}\right) + g - c_y\theta_s = 0 \tag{D1}$$

$$\phi|_{x=0} = 0 \tag{D2}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=1} = -B_x\phi|_{x=1} - B_x\theta_b(y) \tag{D3}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=0} = 0 \tag{D4}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=1} = -B_y\phi|_{y=1} \tag{D5}$$

where $\phi(x,y) \equiv \theta(x,y) - \theta_b(y) = \theta(x,y) - (1 + \gamma - \gamma y^2)\theta_s$. All notations used are consistent with those defined previously. Since Eq. (D2) is a Dirichlet-type boundary condition and Eq. (D4) is a Neumann-type boundary condition, Eq. (D1) can be transformed into an equation of the Fourier integral as follows by multiplying both sides by $\sin\mu x \cos\nu y$ and integrating the equation through the entire domain ($0 \le x \le 1$, $0 \le y \le 1$) with respect to x and y:

$$F(\mu,\nu) = \frac{1}{\mu^2 a_x + \nu^2 a_y}\left[\frac{\sin\nu}{\mu\nu}(1 - \cos\mu)\left(\frac{g}{\theta_s} - c_y\right) - c_x\frac{\sin\mu}{\nu}\left(\sin\nu - B_y\frac{\cos\nu}{\nu} + B_y\frac{\sin\nu}{\nu^2}\right)\right] \tag{D6}$$

where

$$F(\mu,\nu) \equiv \frac{1}{\theta_s}\int_0^1\int_0^1 \phi(x,y)\sin\mu x\cos\nu y\,dx\,dy \tag{D7}$$

and μ and ν are the roots of the following relations:

$$\mu\cos\mu + B_x\sin\mu = 0 \tag{D8}$$

$$\nu\sin\nu - B_y\cos\nu = 0 \tag{D9}$$

Since the value of ν cannot be zero for nonzero Biot numbers, B_y, the function $\phi(x,y)$ can be expanded into Fourier series as follows:

$$\phi(x,y) = \theta_s\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\mu_m x\cos\nu_n y \tag{D10}$$

where C_{mn} are constant coefficients and μ_m and ν_n are the mth and the nth roots of Eqs. (D8) and (D9), respectively. Since the negated values of μ_m and ν_n are also the roots of the corresponding equations, Eq. (D10) can be restricted to the case of positive μ_m and ν_n by redefining the undetermined coefficients C_{mn} as the difference between the coefficients of the terms with positive roots and the corresponding terms with negative roots. Utilizing Eqs. (D8) and (D9), one can show that the functions $\sin\mu_m x$ and $\cos\nu_n y$ are orthogonal functions. In addition, the undetermined coefficients can be evaluated as follows:

$$C_{mn} = \frac{4F(\mu_m, \nu_n) B_x B_y}{(B_x + \cos^2\mu_m)(B_y + \sin^2\nu_n)} \tag{D11}$$

Hence, the solution of Eq. (20) that satisfies the given boundary conditions can be determined as follows:

$$\frac{\theta(x,y)}{\theta_s} = (1+\gamma-\gamma y^2) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\mu_m x \cos\nu_n y \tag{D12}$$

where μ_m and ν_n are positive roots of Eqs. (D8) and (D9), respectively. The total heat transfer rate can be determined by substituting Eq. (D12) into Eqs. (25) and (24) to arrive at the following corresponding expressions:

$$\frac{\dot{Q}}{T_f k_x \theta_s} = \frac{2b}{L}\left(\frac{g}{a_x \theta_s} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \frac{\mu_m}{\nu_n} \sin\nu_n\right) \tag{D13}$$

$$\frac{\dot{Q}}{T_f k_x \theta_s} = \frac{2Lk_y}{bk_x} \times \left[B_y + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\nu_n}{\mu_m} C_{mn} \sin\nu_n (1-\cos\mu_m)\right] - \frac{2b}{L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu_m}{\nu_n} C_{mn} \sin\nu_n \cos\mu_m \tag{D14}$$

Since Eq. (D9) is not the exact solution of the governing equation, the heat transfer evaluated from these equations will not be consistent. The tip temperature at $x=1$ and $y=1$ can be determined by Eq. (D11) as follows:

$$\frac{\theta_t}{\theta_s} = 1 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\mu_m \cos\nu_n \tag{D15}$$

Acknowledgment

The authors would like to thank Professor William H. Sutton of the University of Oklahoma for many helpful discussions.

References

[1]Sfeir, A. A. "The Heat Balance Integral in Steady-State Conduction," *Journal of Heat Transfer, Transactions ASME*, Vol. 98, Aug. 1976, pp. 466–470.

[2]Goodman, T. R., "The Heat Balance Integral and Its Application to Problems Involving a Change of Phase," *Heat Transfer and Fluid Mechanics Institute*, Stanford Univ. Press, 1957; also *Transactions ASME*, Vol. 80, Feb. 1958, pp. 335–342.

[3]Schlichting, H., *Boundary Layer Theory*, 6th ed., McGraw-Hill, New York, 1968, p. 187.

[4]Rayleigh, J. W. S., "On the Theory of Resonance," *Philosophical Transactions of Royal Society of London*, Vol. A161, 1870, pp. 77–118.

[5]Rayleigh, J. W. S., *Theory of Sound*, Vol. I, 2nd ed., Macmillan, London, 1894; reprinted by Dover, New York, 1945, pp. 112–113.

[6]Temple, G. and Bickley, W. G., *Rayleigh's Principle and Its Applications to Engineering*, Oxford Univ Press, New York, 1933; reprinted by Dover, New York, 1945, pp. 94–96.

[7]Schmidt, R., "A Variant of the Rayleigh-Ritz Method," *Industrial Mathematics*, Vol. 31, No. 1, 1981, pp. 37–46.

[8]Bert, C. W., "Application of a Little Known Version of the Rayleigh Technique to Problems of Buckling, Vibration, and Plate Deflection," *Proceedings of the 2nd Conference on Applied Mathematics*, Central Sate Univ., Edmond, OK, April 1986.

[9]Bert, C. W., "Approximate Methods for Analyzing Steady Heat Conduction," *Innovative Numerical Methods in Engineering: Proceedings of 4th International Symposium on Numerical Methods in Engineering*, Springer-Verlag, New York, 1986, pp. 695–700.

[10]Laura, P. A. A., Cortinez, V. H., and Carnicer, R., "Steady-State Temperature Distribution in a Doubly Connected, Orthotropic Region with Heat Generation," *International Journal of Heat and Mass Transfer*, Vol. 30, No. 3, 1987, pp. 610–612.

[11]Arpaci, V. S., *Conduction Heat Transfer*, Addison-Wesley, Reading, MA, 1966, pp. 445–460.

[12]Gerald, C. F., *Applied Numerical Analysis*, Addison-Wesley, Reading, MA, 1970, pp. 2–4.